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C^r-Inclination Theorems for Singularly Perturbed Equations

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1. INTRODUCTION

differential equations

$$x' = \left(= \frac{dx}{d\tau} \right) = f(x, y, \varepsilon) \tag{1.1}$$

$$\varepsilon y' = g(x, y, \varepsilon), \tag{1.2}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, f, g are C^{r+2}, $k > 0$, and $0 \leqslant \varepsilon \ll 1$. Equivalently, with $t = \tau/\varepsilon$, we write (1.1), (1.2) as

$$\dot{x} \left(= \frac{dx}{dt} \right) = \varepsilon f(x, y, \varepsilon) \tag{1.3}$$

$$\dot{y} = g(x, y, \varepsilon). \tag{1.4}$$

A frequently studied problem is to establish dynamically interesting trajectories (like homoclinic, heteroclinic, or satisfying given boundary conditions) for $\varepsilon > 0$ as perturbations of their formal singular $\varepsilon \to 0$ limits. By the latter we understand continuous concatenations of trajectories of the "slow time limit equation"

$$x' = f(x, y, 0) \tag{1.5}$$

$$0 = g(x, y, 0) \tag{1.6}$$

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obtained from (1.1), (1.2) (or, slow arcs), and those of the “fast time limit equation”

$$\dot{x} = 0 \tag{1.7}$$

$$\dot{y} = g(x, y, 0), \tag{1.8}$$

obtained from (1.3), (1.4) (or, layer arcs) for $\varepsilon = 0$.

To deal with trajectories the singular limits of which consist of several slow and fast arcs an effective tool has been developed in [JK] which was called by the authors “exchange lemma” (EL). It is related to the inclination ($\lambda -$) lemma of geometric theory of dynamical systems [PM] but has an additional delicate aspect in the case of higher dimensional slow flows ($m > 1$). Essentially, EL examines up to C^1 accuracy the $\varepsilon \to 0$ asymptotics of the local shape of an invariant manifold intersecting the stable manifold of the $\varepsilon > 0$ perturbation of the slow manifold $g(x, y, 0) = 0$ along a trajectory which stays in the neighborhood of the slow manifold for time $\sim 1/\varepsilon$.

In order to deal with various emerging applications several extensions of EL have been developed.

A survey of the theory as well as of the applications can be found in [J].

The original proofs of the various versions of EL [JK, JKK, T] employ a technique specially developed for this purpose—differential equations for the time development of differential forms. A more elementary proof of the [JKK] version of EL by a shooting argument is presented in [B1].

Bifurcation problems [KSS, B2] exhibit the need to examine this problem in higher than C^1 accuracy as well as in a more global context. In this paper prove a general theorem on an underlying Šilnikov type boundary value problem from which we derive several (“locally”) global C^r versions of the EL.

This approach is different from the differential forms techniques of [JK] as well as the shooting argument of [B1], none of which appears to be suitable for the above extensions. In addition, the proofs do not require any special tools. They rely on the uniform contraction mapping principle in Banach spaces with weighted norms—a device widely used in the geometric theory of differential equations.

The results are hardly surprising, neither is there much novelty in the idea of the proof. In principle, the latter can be found in [D, KSS]. Still, the author feels that it might be useful to document reasonably general theorems tailored to emerging applications with proofs transparent to a wider community of potential readers.

The rest of the paper is organized as follows.

In Section 2 we recall some basic concepts and results from geometric singular perturbation theory. Section 3 contains the formulation of the auxiliary boundary value problem and its proof. Finally, in Section 4 the "locally global" C^r versions of EL are derived from the boundary value problem of Section 3.

2. THE FENICHEL COORDINATES

In this section we recall the necessary material from geometric singular perturbation theory and introduce notation to be used in the rest of the paper. For details and proofs the reader is refereed to [F, S, J].

Let M^0 be a relatively open connected subset of the set $g(x, y, 0)=0$ such that M^0 is normally hyperbolic, i.e., the eigenvalues of $D_y g(x, y, 0)$ are off the imaginary axis for $(x, y) \in \overline{M^0}$. Then, for $(x, y) \in M^0$, y are hyperbolic equilibria of (1.8) with dimensions k, l of the unstable and stable manifolds $W^u(y)$ resp. $W^s(y)$ independent of y. We denote

$$W^j(M^0) = \bigcup_{(x,\, y) \in M^0} W^j(y)$$

for $j = u, s$.

Geometric singular perturbation theory [F] asserts the existence of an $m+1$-dimensional C^{r+1} manifold $\mathbf{M}$ of the (x, y, ε)-space $\mathbb{R}^{m+n+1}$ meeting the hyperplanes $\varepsilon = \text{const}$ transversally such that, for some $\varepsilon_0 > 0$, $M^\varepsilon = \mathbf{M} \cap \{\varepsilon = \text{const}\}$ are C^{r+1} isotopic to M^0 locally invariant manifolds of (1.3), (1.4) for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. There are $k+m+1$, $l+m+1$ C^{r+1} submanifolds $W^u(\mathbf{M})$ resp. $W^s(\mathbf{M})$ of $\mathbb{R}^{m+n+1}$ meeting the hyperplanes $\varepsilon = \text{const}$ transversally such that $W^j(\mathbf{M}) \cap \{\varepsilon = 0\} = W^j(M^0)$ and $W^j(M^\varepsilon) = W^j(\mathbf{M}) \cap \{\varepsilon = \text{const}\}$ are locally invariant manifolds of (1.3), (1.4) containing M^ε. The trajectories in $W^u(M^\varepsilon)$ resp. $W^s(M^\varepsilon)$ approach M^ε uniformly exponentially for t decreasing resp. increasing.

Locally in a uniform neighborhood of M^ε a C^r coordinate system generated by the "Fenichel fibering" in $\mathbb{R}^{m+n}$ can be introduced in which (1.3), (1.4) is represented by a system of equations

$$\dot{a} = A(a, b, x, \varepsilon)\, a \tag{2.1}$$

$$\dot{b} = B(a, b, x, \varepsilon)\, b \tag{2.2}$$

$$\dot{x} = \varepsilon(X(x, \varepsilon) + C(a, b, x, \varepsilon)\, ab), \tag{2.3}$$

where $\dim a = k$, $\dim b = l$, $\dim x = m$, $M^\varepsilon = \{a = 0, b = 0\}$, $W^u(M^\varepsilon) = \{b = 0\}$, $W^s(M^\varepsilon) = \{a = 0\}$, A, B, X, C are C^r and, for fixed $\bar{a}$, $\bar{b}$, $(a, b) \mapsto C(\bar{a}, \bar{b}, x, \varepsilon)\, ab$ is bilinear. One has

$$\langle a, Aa\rangle \geqslant 4\alpha\, |a|^2 \tag{2.4}$$

$$\langle b, Bb\rangle \leqslant -4\beta\, |b|^2, \tag{2.5}$$

where $\alpha, \beta > 0$, (cf. [J, (2.2), Lemma 1]).

3. THE BOUNDARY VALUE PROBLEM

In this section we consider the system (2.1)–(2.3) with A, B, C, X being C^r and A, B satisfying (2.4)–(2.5) in the compact region

$$\Omega = \{|a| \leqslant \bar{\Delta}, |b| \leqslant \bar{\Delta}, x \in \bar{M}, 0 \leqslant \varepsilon \leqslant \bar{\varepsilon}\},$$

M an open subset of $\mathbb{R}^m$ with compact closure (note that we can identify the sets M^ε for $\varepsilon \geqslant 0$ small because of their isotopy).

By φ_t^ε we denote the flow of (2.1)–(2.3), by ρ_τ^ε the flow of the "reduced" equation

$$x' = X(x, \varepsilon). \tag{3.1}$$

For given x^0, it will be convenient to replace Eq. (2.3) by the equivalent equation

$$\dot{y} = h(a, b, y, x^0, t, \varepsilon) \tag{3.2}$$

for $y = x - \rho_{\varepsilon t}(x^0)$, where

$$h(a, b, y, x^0, t, \varepsilon)$$
$$= \varepsilon[X(\rho_{\varepsilon t}(x^0) + y, \varepsilon) - X(\rho_{\varepsilon t}(x_0), \varepsilon) + C(a, b, \rho_{\varepsilon t}(x^0) + y, \varepsilon)\, ab]. \tag{3.3}$$

When working with fixed ε, for simplicity we will occasionally omit it as argument, sub- or superscript. Boundedness and all estimates will be understood to be uniform for ε sufficiently small.

We will occasionally aggregate the variables by denoting $u = (a, b)$, $w = (a, b, x)$. For $j = a, b, u$, etc., we denote by P_j the natural projection of w to the j-component.

By $\Phi(t, t_0)(= \Phi(t, t_0, x^0))$ we denote the fundamental transition matrix of the linearization of (2.1), (2.2) along the trajectory $\rho_{\varepsilon t}(x^0)$ of (3.1). That is, $\Phi(t, t_0)$ is the fundamental matrix of the linear system of equations

$$\dot{a} = A(0, 0, \rho_{\varepsilon t}(x^0))\, a \tag{3.4}$$

$$\dot{b} = B(0, 0, \rho_{\varepsilon t}(x^0))\, b, \tag{3.5}$$

satisfying $\Phi(t_0, t_0) = I$. Obviously, $\Phi(t, t_0)$ commutes with P_a, P_b and, by (2.4), (2.5),

$$|\Phi(t, t_0)\, P_a| \leqslant e^{2\alpha(t-t_0)} \qquad \text{for} \quad t \leqslant t_0 \tag{3.6}$$

$$|\Phi(t, t_0)\, P_b| \leqslant e^{-2\beta(t-t_0)} \qquad \text{for} \quad t \geqslant t_0. \tag{3.7}$$

There is a $\Theta > 0$ such that

$$|\Phi(t, t_0)\, P_a| \leqslant 2 \qquad \text{for} \quad 0 \leqslant t_0 \leqslant \bar{\tau}/\varepsilon,\ t_0 \leqslant t \leqslant t_0 + \Theta. \tag{3.8}$$

Finally, for $\underline{\tau} \leqslant \varepsilon T_0 < \bar{\tau}$ we denote by $\mathscr{C} = \mathscr{C}(T_0)$ the space of continuous functions $w(\cdot) = (a(\cdot), b(\cdot), y(\cdot))$ on $[0, T_0]$ with the norm

$$\|w\| = \|a\| + \|b\| + \|y\|$$

$$\|a\| = \sup_{0 \leqslant t \leqslant T_0} e^{\alpha(T_0 - t)}\, |a(t)|$$

$$\|b\| = \sup_{0 \leqslant t \leqslant T_0} e^{\beta t}\, |b(t)|$$

$$\|y\| = \sup_{0 \leqslant t \leqslant T_0} e^{-\kappa\varepsilon t}\, |y(t)|,$$

κ to be specified later.

Note that

$$|a(t)| \leqslant e^{-\alpha(T_0 - t)}\, \|a\| \leqslant \|a\| \tag{3.9}$$

$$|b(t)| \leqslant e^{-\beta t}\, \|b\| \leqslant \|b\| \tag{3.10}$$

$$|x(t)| \leqslant e^{\varepsilon\kappa t}\, \|x\| \leqslant e^{\kappa\bar{\tau}}\, \|x\| \tag{3.11}$$

for $0 \leqslant t \leqslant T_0$.

THEOREM 3.1. *Let* $\mathbf{J}$ *be a submanifold of* $\mathbf{M} = M \times \mathbb{R}$ *meeting the hyperplanes* $\varepsilon = \text{const}$ *transversally; by* J^ε *denote the* ε*-section of* $\mathbf{J}$, $0 \leqslant \varepsilon \leqslant \bar{\varepsilon}$. *Assume* $\rho^0_\tau(J^0) \subset M$ *for* $0 \leqslant \tau \leqslant \bar{\tau}$. *Given* $\underline{\tau} \in (0, \bar{\tau})$, *there exist* ε_0, $\Delta > 0$, $d > 0$ *such that for all* $T_0 \in [\underline{\tau}/\varepsilon, \bar{\tau}/\varepsilon]$, $|b^0| \leqslant \Delta$, $|a^1| \leqslant \Delta$, $x^0 \in J^\varepsilon$ *and* $T_0 - \Theta \leqslant T < T_0$ *there exists a unique solution* $\hat{w}(\cdot) = \Psi(a^1, b^0, x^0, T) \in \mathscr{C}$ *of the system of Eqs.* (2.1), (2.3), (3.2) *satisfying*

$$a(T) = a^1, \qquad b(0) = b^0, \qquad y(0) = 0. \tag{3.12}$$

The function $\Psi : \{|a^1| \leqslant \Delta\} \times \{|b^0| \leqslant \Delta\} \times J^\varepsilon \times (T_0 - \Theta, T_0) \to \mathscr{C}$ *is* C^r *with bounded derivatives; one has*

$$|(D^j P_b \Psi)(T)| = O(e^{-q/\varepsilon}) \tag{3.13}$$

$$|(D^j P_a \Psi)(0)| = O(e^{-q/\varepsilon}) \tag{3.14}$$

and

$$\|D^j P_y \Psi\| = O(e^{-q/\varepsilon}) \tag{3.15}$$

for some $q > 0$ *and all* $0 \leqslant j \leqslant r$.

3.2. *Remark.* The application of Theorem 3.1 in Section 4 requires differentiability of the solution of the boundary value problem (2.1), (2.2), (3.2), (3.12) with respect to T. We locate the problem into an extended interval $[0, T_0]$ instead of the interval $[0, T]$ of the boundary value problem in order to avoid the change of the function space with changing T. Alternatively, one could fix the interval by rescaling time as in [KSS]. This, however, makes differentiation of higher order complicated.

3.3. *Remark.* Since $X(\cdot, \varepsilon)$ uniformly in $\bar{M}$ tends to $X(\cdot, 0)$ in a C^r way, (3.15) implies

$$|D^j(P_x \Psi(t) - \rho^0_{\varepsilon t}(x^0))| = O(\varepsilon)$$

for $0 \leqslant j < r$ and

$$|D^r(P_x \Psi(t) - \rho^0_{\varepsilon t}(x^0))| \to 0 \qquad \text{for} \quad \varepsilon \to 0$$

for $0 \leqslant t \leqslant T$.

Proof. For a fixed x^0 we rewrite Eqs. (2.1), (2.2) as

$$\dot{a} = A(0, 0, \rho_{\varepsilon t}(x^0))\, a + f(w, x^0, t)$$

$$\dot{b} = B(0, 0, \rho_{\varepsilon t}(x^0))\, b + g(w, x^0, t),$$

where

$$f(u, x, x^0, t) = [A(u, x) - A(0, \rho_{\varepsilon t}(x^0))]\, a \tag{3.16}$$

$$g(u, x, x^0, t) = [B(u, x) - B(0, \rho_{\varepsilon t}(x^0))]\, b. \tag{3.17}$$

Obviously, all the derivatives of f, g are bounded and for $0 \leqslant j_a + j_b + j_y + j_{x^0} \leqslant r$ we have

$$|D_a^{j_a} D_b^{j_b} D_y^{j_y} D_{x^0}^{j_{x^0}} f| = \begin{cases} O(|a|) & \text{if } \; j_a = 0 \\ O(|w|) & \text{if } \; j_a = 1, \; j_b = j_y = j_{x_0} = 0, \end{cases} \tag{3.18}$$

$$|D_a^{j_a} D_b^{j_b} D_y^{j_y} D_{x^0}^{j_{x^0}} g| = \begin{cases} O(|b|) & \text{if } \; j_b = 0 \\ O(|w|) & \text{if } \; j_b = 1, \; j_a = j_y = j_{x_0} = 0, \end{cases} \tag{3.19}$$

Similarly, all the derivatives of h are bounded and

$$|D_{x^0}^{j} h(w, x^0, t)| = O(|y| + |a| \; |b|) \tag{3.20}$$

for $0 < j \leqslant r$, uniformly in Ω for $0 \leqslant \varepsilon \leqslant \bar{\varepsilon}$.

A function $w(t) = (u(t), y(t))$, $u(t) = (a(t), b(t))$ is a solution of (2.1), (2.2), (3.2), (3.12) if and only if

$$\begin{aligned} u(t) = {} & \Phi(t, T, x^0) \, a^1 + \int_T^t \Phi(t, s, x^0) \, f(w(s), x^0, s) \, ds \\ & + \Phi(t, 0, x^0) \, b^0 + \int_0^t \Phi(t, s, x^0) \, g(w(s), x^0, s) \, ds \end{aligned} \tag{3.21}$$

$$y(t) = \varepsilon \int_0^t h(w(s), x^0, s) \, ds. \tag{3.22}$$

for $0 \leqslant t \leqslant T_0$ (in (3.21) we identify points $a \in \mathbb{R}^k$, $b \in \mathbb{R}^l$ with $(a, 0)$, $(0, b) \in \mathbb{R}^{k+l}$, respectively). Denote

$$\mathscr{D}_\Delta = \{ w \in \mathscr{C} : \|w\| \leqslant 4\Delta \}.$$

For $w \in \mathscr{D}_\Delta$ we denote by $\mathscr{F} : \mathscr{D}_\Delta \to \mathscr{C}$ the mapping assigning to $w \in \mathscr{D}_\Delta$ the vector of the right hand sides of (3.21), (3.22). We prove that for ε_0, Δ sufficiently small, κ sufficiently large and $0 < \varepsilon \leqslant \varepsilon_0$, $\mathscr{F} : \mathscr{D}_\Delta \to \mathscr{D}_\Delta$ is a contraction with contraction quotient 1/4. This is obviously true if

$$\|\mathscr{F}(0)\| \leqslant 3\Delta, \tag{3.23}$$

$\mathscr{F}$ is differentiable in w and

$$\|D_w \mathscr{F}\| = O(\Delta) \tag{3.24}$$

for $w \in \mathscr{D}_\Delta$ and $0 < \varepsilon \leqslant \varepsilon_0$.

As a composition of a nonlinear Nemytski operator between functional Banach spaces with supremum norms and bounded linear maps in such spaces, $\mathscr{F}$ is C^r-differentiable. Furthermore, its derivatives are obtained by differentiating formally the generating functions f, g, h with respect to the finite dimensional variables a, b, y.

By (3.6)–(3.8), (3.18)–(3.20), we have

$$\|P_u\mathscr{F}(0)\| \leqslant 2\,|a^1| + |b^0| \leqslant 3\Delta, \qquad P_y\mathscr{F}(0) = 0,$$

which implies (3.23).

By (3.6), (3.8), (3.9)–(3.11), for $\delta w = (\delta a, \delta b, \delta y) \in \mathscr{C}$ and $T \in [T_0 - \Theta, T_0]$ we have

$$\begin{aligned}
&|e^{\alpha(T_0-t)} P_a D_w \mathscr{F}(w)\,\delta w| \\
&\quad = \left| e^{\alpha(T_0-t)} \int_T^t \Phi(t,s)\, P_a\, D_w f(w(s), x^0, s)\,\delta w(s)\, ds \right| \\
&\quad \leqslant \int_t^T e^{\alpha(T_0-t)} e^{2\alpha(t-s)}\, |D_a f(w(s), x^0, s)\,\delta a(s) \\
&\qquad + D_b f(w(s), x^0, s)\,\delta b(s) + D_y f(w(s), x^0, s)\,\delta y(s)|\, ds \\
&\quad = \int_t^T e^{\alpha(t-s)} [\, O(|w(s)|)\, |e^{\alpha(T_0-s)} \delta a(s)| \\
&\qquad + O(|e^{\alpha(T_0-s)} a(s)|)(|\delta b(s)| + |\delta y(s)|)]\, ds \\
&\quad = O((1 + e^{\kappa\varepsilon T_0})\,\Delta)\,\|\delta a\| + O(\Delta)(\|\delta b\| + e^{\kappa\varepsilon T_0}\,\|\delta y\|) \\
&\quad = O(e^{\kappa\bar{\tau}}\Delta)\,\|\delta w\|
\end{aligned} \tag{3.25}$$

for $t \leqslant T$ and

$$\left| e^{\alpha(T_0-t)} \int_T^t \Phi(t,s)\, P_a\, D_w f(w(s), x^0, s)\,\delta w(s)\, ds \right| \leqslant O(2\Theta e^{\kappa\varepsilon\bar{\tau}}\Delta)(\|\delta w\|) \tag{3.26}$$

for $T \leqslant t \leqslant T_0$. Summarizing (3.25) and (3.26) we obtain

$$\|D_w P_a \mathscr{F}(w)\| = O(\Delta). \tag{3.27}$$

Similarly, one obtains

$$\|D_w P_b \mathscr{F}(w)\| = O(\Delta), \tag{3.28}$$

both for $w \in \mathscr{D}_\Delta$.

Further, by (3.9), (3.10) we have

$$\begin{aligned}
&|e^{-\kappa\varepsilon t}(D_a P_y \mathscr{F}(w)\,\delta a)(t)| \\
&\quad = \varepsilon e^{-\kappa\varepsilon t} \int_0^t |D_a C(w(s))\,\delta a(s)\,a(s)\,b(s)\,ds + C(w(s))\,\delta a(s)\,b(s)|\,ds \\
&\quad = \varepsilon \Delta O\left(\int_0^{T_0} e^{-\alpha(T_0-s)} e^{-\beta s}\,ds\right) \|\delta a\| \\
&\quad = \varepsilon \Delta O\left(e^{-\alpha T_0} \int_0^{T_0} e^{(\alpha-\beta)\,s}\,ds\right) \|\delta a\| \\
&\quad = O(\varepsilon\Delta)\,\|\delta a\|,
\end{aligned}$$

hence

$$\|D_a P_y \mathscr{F}(w)\| = O(\varepsilon\Delta). \tag{3.29}$$

Similarly, we obtain

$$\|D_b P_y \mathscr{F}(w)\| = O(\varepsilon\Delta). \tag{3.30}$$

Finally, by (3.11) we have

$$|e^{-\kappa\varepsilon t}(D_y P_y \mathscr{F}(w)\,\delta y)(t)| \leqslant \varepsilon \left| \int_0^t e^{-\kappa\varepsilon(t-s)} e^{-\kappa\varepsilon s}\,\delta y(s)\,ds \right| = O\left(\frac{1}{\kappa}\right) \|\delta y\|,$$

hence

$$\|D_y P_y \mathscr{F}(w)\| = O\left(\frac{1}{\kappa}\right). \tag{3.31}$$

From (3.27)–(3.31) it follows that (3.23), (3.24) hold and, hence, for $\Delta > 0$ sufficiently small and κ sufficiently large, $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ is a contraction with quotient 1/4 for $0 < \varepsilon \leqslant \varepsilon_0$ sufficiently small.

By the uniform contraction principle, for $\Delta > 0$ and $0 < \varepsilon \leqslant \varepsilon_0$ sufficiently small, $\mathscr{F}$ has a unique fixed point in $\mathscr{D}_\Delta$. Since, in addition to C^r differentiability with respect to w, $\mathscr{F}$ is obviously differentiable with respect to a^1, b^0, T, x^0, by the uniform contraction principle, the fixed point $\hat{w} = \Psi(a^1, b^0, x^0, T)$ is C^r-differentiable as a function of a^1, b^0, x^0, T. We now estimate its derivatives.

To estimate the derivatives of Ψ we recall [CH] that they are obtained by formal differentiation of the fixed point identity $\Psi(p) = \mathscr{F}(\Psi(p), p)$, where p stands for the vector (a^1, b^0, x^0, T). Denoting $G : p \mapsto (\Psi(p), p)$ and using the chain rule formula [L], for $j > 0$ we obtain

$$\begin{aligned} D^j\Psi(p)\,\delta p_1 \cdots \delta p_j &= D^j_p \mathscr{F}(\Psi(p), p)\,\delta p_1 \cdots \delta p_j \\ &= \sum_{i=1}^{j} \sum_{[l_1, ..., l_i]} D^i\mathscr{F}(G(p))\, D^{|l_1|}G(p)\, l_1 \cdots D^{|l_i|}G(p)\, l_i \\ &= D_w\mathscr{F}(G(p))\, D^j\Psi(p)\,\delta p_1 \cdots \delta p_j + R, \end{aligned} \tag{3.32}$$

where

$$R = \sum_{i=2}^{j} \sum_{[l_1, ..., l_i]} D^i\mathscr{F}(G(p))\, D^{|l_1|}G(p)\, l_1 \cdots D^{|l_i|}G(p)\, l_i + \gamma D_p\mathscr{F}(G(p))\,\delta p_1, \tag{3.33}$$

$[l_1, ..., l_i]$ runs through all i-member partitions of the j-tuple $\delta p_1, \cdots \delta p_j$, $|l_v|$ is the cardinality of the partition member l_v, $\gamma = 1$ for $j = 1$, and $\gamma = 0$ otherwise. From (3.33) it follows

$$\|R\| \leqslant \Pi_j\ \|\delta p_1\|, ..., \|\delta p_j\|, \tag{3.34}$$

where Π_j is a polynomial of jth order of the variables $\|D^v\Psi(p)\|$, $|v| = 1, ..., j-1$, its coefficients being derivatives of $\mathscr{F}$ in w and p up to order j.

Since $\|D_w\mathscr{F}(\Psi(p), p)\| \leqslant 1/4$, from (3.32) it follows

$$\|D^j\Psi(p)\| \leqslant (1 - \|D\mathscr{F}(\Psi(p), p)\|)^{-1}\, \Pi_j \leqslant 2\Pi_j. \tag{3.35}$$

By virtue of (3.9), (3.10), in order to establish the estimates (3.13), (3.14), it is sufficient to prove boundedness of $D^jP_a\Psi$, $D^jP_b\Psi$. To this end we observe that the latter follows by induction once we prove boundedness of the derivatives of $\mathscr{F}$ in w and p, which are the coefficients of the polynomials Π_j.

Since $P_a\mathscr{F}$ does not depend on b^0, for $0 \leqslant j = j_w + j_{a^1} + j_{b_0} + j_{x^0} + j_T \leqslant r$ we have

$$\begin{aligned} &D^{j_w}_w D^{j_{a^1}}_{a^1} D^{j_{b^0}}_{b^0} D^{j_{x^0}}_{x^0} D^{j_T}_T P_a\mathscr{F}(w, p)(t) \\ &\quad = \delta(j_{b^0})\,\delta(j_{a^1} - 1)\,\delta(j_w)\, D^{j_{x^0}}_{x^0} D^{j_T}_T \Phi(t, T, x^0) \\ &\qquad + \delta(j_{b^0})\,\delta(j_{a^1}) \Big\{ \delta(j_w)\, D^{j_{x^0}}_{x_0} D^{j_T}_T \Phi(t, T, x^0) a^1 \\ &\qquad - [1 - \delta(j_T)]\, D^{j_w}_w D^{j_{x^0}}_{x^0} D^{j_T - 1}_T [\Phi(t, T, x^0)\, f(w(T), x^0, T)] \\ &\qquad + \delta(j_T) \int_T^t D^{j_w}_w D^{j_{x^0}}_{x^0} [\Phi(t, s, x^0)\, f(w(s), x^0, s)]\, ds \Big\}, \end{aligned} \tag{3.36}$$

where $\delta(0)=1$, $\delta(n)=0$ for $n\neq 0$. By differentiating the generating equations

$$\frac{d}{dt}\Phi(t,T)=A(0,0,\rho_{\varepsilon t}(x^0))\,\Phi(t,T),\qquad \Phi(T,T)=I$$

for Φ with respect to T and x^0 one verifies readily that all the derivatives of Φ up to order r with respect to x^0 and T are $O(e^{-2\alpha(T-t)})$ for $w\in\mathscr{D}_\Delta$ and $0<\varepsilon\leqslant\varepsilon_0$.

Therefore, $|e^{\alpha(T_0-t)}D^jP_a\mathscr{F}(w,p)(t)|$ and, consequently, $\|D^jP_a\mathscr{F}(w,p)\|$ is bounded (to estimate the integral term we proceed as in (3.25), (3.26) using (3.9), (3.10)).

Hence, $\|D^jP_a\mathscr{F}(w,p)\|$ is bounded for $w\in\mathscr{D}_\Delta$ and $0<\varepsilon\leqslant\varepsilon_0$. For similar reasons, so is $\|D^jP_b\mathscr{F}(w,p)\|$. This completes the proof of (3.13), (3.14).

Since $P_y\mathscr{F}$ does not depend on a^1, b^0, T, we have

$$\begin{aligned}
&D_w^{j_w}D_{a^1}^{j_{a^1}}D_{b^0}^{j_{b^0}}D_{x^0}^{j_{x^0}}D_T^{j_T}P_y\mathscr{F}(w,p)\\
&\qquad=0\qquad \text{if}\quad j_{a^1}>0 \text{ or } j_{b^0}>0 \text{ or } j_T>0,\\
&\qquad=\varepsilon\int_0^t D_w^{j_w}D_{x^0}^{j_{x^0}}h(w(s),x^0,s)\,ds\\
&\qquad=O(\bar{\tau}e^{\kappa\bar{\tau}k})\qquad \text{otherwise}
\end{aligned}$$

because of (3.11). This proves boundedness of the derivatives of $P_y\mathscr{F}$.

To complete the proof, it remains to prove (3.15). We first prove (3.15) for $j=0$, i.e.,

$$\|\hat{y}\|=O(e^{-q/\varepsilon})\tag{3.37}$$

for a suitable q. Applying the Leibniz–Newton formula to the first two terms of (3.3) we obtain

$$|\hat{y}(t)|=O(\varepsilon)\int_0^t|\hat{y}(s)|\,ds+O(\varepsilon\Delta^2)\int_0^t e^{\alpha(s-T_0)}e^{-\beta s}\,ds.$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}
|\hat{y}(t)|&=\varepsilon\Delta^2\int_0^t e^{O(\varepsilon)(t-s)}O(e^{\alpha(s-T_0)}e^{-\beta s})\,ds\\
&=O(\varepsilon\Delta^2\max\{e^{-\beta\underline{\tau}/\varepsilon},e^{(-\alpha+O(\varepsilon))\,\underline{\tau}/\varepsilon}\}),
\end{aligned}$$

hence (3.37) holds for $T\geqslant\underline{\tau}/\varepsilon$ provided $\varepsilon\leqslant\varepsilon_0$ and $O(\varepsilon_0)<\alpha$.

Since h does not depend on a^1, b^0, T, from (3.37) it follows that for j as in (3.36) we have

$$\begin{aligned} &|D_{a^1}^{j_{a^1}} D_{b^0}^{j_{b^0}} D_{x^0}^{j_{x^0}} D_T^{j_T} h(t, \hat{w}(t), t)| \\ &\quad \leqslant |D_{a^1}^{j_{a^1}} D_{b^0}^{j_{b^0}} D_{x^0}^{j_{x^0}} D_T^{j_T} (X(\rho_{\varepsilon t}(x^0) + \hat{y}(t)) - X(\rho_{\varepsilon t}(x^0))| \\ &\quad = \begin{cases} O(e^{-q/\varepsilon}) & \text{if} \quad j_{a^1} = j_{b^0} = j_T = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.38}$$

for some $q > 0$.

Further, we have

$$\begin{aligned} &\|D_a P_y \mathscr{F}(p, \Psi(p))\, \delta a\| \\ &\quad \leqslant \varepsilon \int_0^T [D_a C(w(s))\, \delta a(s)\, a(s)\, b(s) + C(w(s))\, \delta a(s)\, b(s)\, ds] \\ &\quad = O(e^{-\alpha T} + e^{-\beta T}) \|\delta a\| = O(e^{-q/\varepsilon}) \|\delta a\| \end{aligned}$$

with $q = \min\{\alpha \underline{\tau}/\varepsilon, \beta \underline{\tau}/\varepsilon\}$. Since $D_b P_y \mathscr{F}$ can be estimated similarly, one has

$$\|D_u P_y \mathscr{F}(w, p)\| = O(e^{-q/\varepsilon}) \tag{3.39}$$

with the same q. To complete the proof we need the following

LEMMA 3.4. *Let X_1, X_2 be Banach spaces with norms denoted by $|\cdot|$, $C_{ij} : X_i \to X_j$ be bounded linear operators, $i = 1, 2$ and*

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} : X_1 \times X_2 \to X_1 \times X_2$$

satisfies $|C| < 1$. Then, for $x \in X_1 \times X_2$ we have

$$|P_2(I - C)^{-1} x| \leqslant |P_2 x| + |P_2 C|\, |(1 - |C|)^{-1}|\, |x|,$$

where P_2 is the natural projection of $X_1 \times X_2$ onto X_2.

Proof. The lemma follows immediately from the identity

$$(I - C)^{-1} = I + C(I - C)^{-1}$$

and the series representation of $(I - C)^{-1}$. ∎

We apply this lemma with X_1, X_2 the $u = (a, b)$- resp. y-subspace of $\mathscr{C}$ and $C = D\mathscr{F}(\Psi(p), p)$. By (3.32) and the lemma, we have

$$\begin{aligned}\|D^j P_y \Psi(p)\,\delta p_1 \cdots \delta p_j\| &= \|P_y(I - D_w \mathscr{F}(G(p))^{-1} R\| \\ &\leqslant \|P_y R\| + 2\,\|P_y D\mathscr{F}(G(p))\|\ \|R\| \\ &\leqslant \sum_{i=2}^{j} \sum_{[l_1, \ldots, l_i]} [H_{i1} + H_{i2}] + H_0, \end{aligned} \tag{3.40}$$

where

$$H_{i1} = (1 + 2\,\|D_y P_y D\mathscr{F}(G(p))\|)\ \|D^i P_y \mathscr{F}(G(p))\, D^{|l_1|} G(p)\, l_1 \cdots D^{|l_i|} G(p)\, l_i\|$$
$$H_{i2} = 2\,\|D_u P_y D\mathscr{F}(G(p))\|\ \|D^i P_u \mathscr{F}(G(p))\, D^{|l_1|} G(p)\, l_1 \cdots D^{|l_i|} G(p)\, l_i\|$$

and

$$H_0 = \|\gamma D_p P_y \mathscr{F}(G(p))\|\ \|\delta p_1\| = O(e^{-q/\varepsilon})\ \|\delta p_1\|$$

by (3.38), γ as in (3.33).

To extend (3.15) to $j \neq 0$ we proceed by induction. Assume that (3.15) holds up to $j-1$. Note that every term $D^i P_y \mathscr{F} D^{|l_1|} G(p)\, l_1 \cdots D^{|l_i|} G(p)\, l_i$ either contains a factor $D^\nu \Psi$ with $\nu < k$, or else is a differential with respect to the parameter p only. Hence H_{i1} is $O(e^{-q/\varepsilon})$ with a suitable $q > 0$ by the induction hypothesis (3.38) and the boundedness of the derivatives of $\mathscr{F}$. The term H_{i2} is $O(e^{-q/\varepsilon})$ because of boundedness of the derivatives and (3.39). This completes the proof of the theorem. ∎

Assume now that **J** is as in Theorem 3.1. Then, J^ε is a C^r submanifold of M for $0 \leqslant \varepsilon \leqslant \varepsilon_0$ small. Assume that

$$X(x^0, \varepsilon) \notin T_{x^0} J^\varepsilon \tag{3.41}$$

for $0 < \varepsilon \leqslant \varepsilon_0$ and, moreover,

$$|X(x^0, \varepsilon)| \geqslant C\varepsilon^\lambda \qquad \text{and} \qquad \text{angle}(X(x^0, \varepsilon), T_{x^0} J^\varepsilon) \geqslant C\varepsilon^\lambda \tag{3.42}$$

for some $C > 0$, $\lambda > 0$. Further, assume that the map $J^\varepsilon \times [\underline{\tau}, \bar{\tau}] \to \mathbb{R}^m$ given by $(x^0, \tau) \mapsto \rho^\varepsilon_\tau(x^0)$ is injective for $0 \leqslant \varepsilon \leqslant \varepsilon_0$ sufficiently small as well. From this injectivity and (3.41) it follows that the set

$$N^\varepsilon = \{x = \varphi^\varepsilon_t(x^0) : x^0 \in J^\varepsilon, \underline{\tau} < \varepsilon t < \bar{\tau}\} \tag{3.43}$$

is an immersed submanifold of M. Indeed, the Jacobi matrix of the map $J^\varepsilon \times (\underline{\tau}, \bar{\tau}) \to M$ at (x^0, τ) given by $(x^0, \tau) \mapsto \rho_\tau(x^0)$ is $Y(\tau)$, where $Y(\tau)$ is the solution of the $m \times (\dim J^\varepsilon + 1)$-matrix equation

$$\dot{Y} = DX(\rho_{\varepsilon t}(x^0), \varepsilon)\ Y \tag{3.44}$$

with initial condition

$$Y(0) = (V, X(x^0, \varepsilon)), \tag{3.45}$$

V being the matrix of some unit base column vectors of $T_{x^0}J^\varepsilon$. Thus, $Y(0)$ and, consequently, $Y(\tau)$, has maximal rank because of (3.41).

Since the map $(x^0, \tau) \mapsto \rho_\tau(x^0)$ is injective on $J^\varepsilon \times [\underline{\tau}, \bar{\tau}]$ the topology on N^ε induced by this map coincides with the relative one. Therefore, N^ε is an imbedded submanifold of M. This means that N^ε can be locally (in the relative topology) reparametrized by some natural coordinates compatible with its imbedding (like, e.g., the coordinates of its orthogonal projections on the tangent spaces of some of its points). Hence, Ψ generates a well defined C^r function $\tilde{\Psi}$ of a^1, b^0 and $x \in N^\varepsilon$ by

$$\tilde{\Psi}(a^1, b^0, x) = \Psi(a^1, b^0, i(x)), \tag{3.46}$$

where $i : x \mapsto (x^0(x), \tau(x))$ is the inverse of $(x^0, \tau) \mapsto \rho_\tau(x^0)$. From (3.42) it follows by the inverse function theorem that the derivatives of i grow at most with some power of ε^{-1} for $\varepsilon \to 0$. Hence, we have

3.5. Corollary. *Assume that* (3.41), (3.42) *hold and that the map* $(x^0, \tau) \mapsto \rho_\tau(x^0)$ *is injective on* $J \times [\underline{\tau}, \bar{\tau}]$. *Then, for* $0 < \varepsilon \leqslant \varepsilon^0$ *sufficiently small,* N^ε *defined by* (3.43) *is an imbedded submanifold of* M. *The map* Ψ *generates a* C^r *map* $\tilde{\Psi}$ *on* $\{|a^1| \leqslant \Delta\} \times \{b^0 \leqslant \Delta\} \times \rho_{(\underline{\tau}, \bar{\tau})} J^\varepsilon$ *the derivatives of which satisfy the estimates* (3.13), (3.14) *with* Ψ *replaced by* $\tilde{\Psi}$ *and*

$$|D^j P_y \tilde{\Psi}| = O(e^{-q/\varepsilon}). \tag{3.47}$$

3.6. *Remark.* An obvious local (in (x^0, τ)) version of Corollary 3.4 holds without the injectivity assumption. That is, if (3.41), (3.42) hold then $\tilde{\Psi}$ is well defined and satisfies (3.13), (3.14) and (3.47) locally at an individual point (x^0, τ). This local point of view is adopted in [J, Lemma 7].

4. INCLINATION THEOREMS

In this section we make use of Theorem 3.1 (and Corollary 3.4) to obtain "locally global" C^r extensions of the Exchange Lemmas [J]. By saying locally global we mean that their conclusions concern the entire invariant manifold of the local flow φ_t^ε rather than a neighborhood of an individual point. In this respect they are closer in spirit to the inclination lemma [PM]; this is why we call them "inclination theorems." We keep the terminology and notation of Section 3 without further notice.

By an entry manifold we understand a set

$$\mathbf{\Sigma} = \{(a, b, x, \varepsilon) : b = \boldsymbol{\sigma}(a, x, \varepsilon), |a| \leqslant \Delta, x \in \mathbf{J}, 0 \leqslant \varepsilon \leqslant \varepsilon_0\}. \tag{4.1}$$

Along the lines of Section 3, we denote by J^ε, Σ^ε, σ^ε the ε-sections of $\mathbf{J}$, $\mathbf{\Sigma}$, $\boldsymbol{\sigma}$, respectively, and, occasionally, when working with a fixed ε, we drop it as argument, sub-, or superscript.

In Section 3 we established existence of an invariant manifold of solutions satisfying boundary condition on the a- and b-components at different ends of the time interval. In addition. we derived exponential estimates for the derivatives of the function generating the manifold as its graph. In this section we exploit these results to obtain similar information about an invariant manifold given by a condition tying the a-, b-, and y-components at one side of the time interval.

In the context of the exchange lemmas, the points of Σ^ε are viewed as entry points of trajectories of an invariant manifold of (1.3), (1.4) into a Fenichel coordinated neighborhood of a branch of the slow manifold. The theorem below corresponds to a version of the "$(k+\sigma)$-exchange lemma" in which the points of the invariant manifold are parameterized by the entry point of the trajectory, the unstable variable and the time elapsed from the entry [J, Lemma 7].

THEOREM 4.1. *Let Σ^ε be defined by* (4.1) *with*

$$|\sigma^\varepsilon(a, x)| \leqslant \Delta, \qquad |D^j\sigma(a, x)| \leqslant O(\varepsilon^{-\lambda}) \tag{4.2}$$

for $j \leqslant r$ and Δ sufficiently small. Then, there exists a function $s^\varepsilon : \{(a, x^0, T) : |a| \leqslant \Delta; x^0 \in J^\varepsilon, \underline{\tau} < \varepsilon T < \bar{\tau}\} \to \mathbb{R}^{l+m}$ such that

$$(a, b, x) = \varphi^\varepsilon_T(a^0, \sigma^\varepsilon(a^0, x^0), x^0) \tag{4.3}$$

for some $|a^0| \leqslant \Delta$, $x^0 \in J^\varepsilon$ if and only if

$$(b, x) = s^\varepsilon(a, x^0, T). \tag{4.4}$$

Furthermore,

$$|D^j P_b s^\varepsilon(a, x^0, T)| = O(e^{-q/\varepsilon}) \tag{4.5}$$

$$|D^j P_x(s^\varepsilon(a, x^0, T) - \rho^\varepsilon_{\varepsilon T}(x^0))| = O(e^{-q/\varepsilon}) \tag{4.6}$$

for $0 \leqslant j \leqslant r$ and some $q > 0$.

4.2. *Remark.* Let us stress again that, in Theorem 4.1, we understand φ to be the local flow in $M \times \{|a| \leqslant \Delta\} \times \{|b| \leqslant \Delta\}$ (the constraints on b can be dropped). That is, (4.4) characterizes only points on those trajectories through Σ^{ε} which do not leave the set $|a| \leqslant \Delta$.

4.3. *Remark.* Note that (4.2) implies that Σ^{ε} meets the stable manifold $a=0$ transversally, transversality possibly decaying polynomially with $\varepsilon \to 0$.

4.4. *Remark.* The assumption of $\mathbf{\Sigma}$ being a manifold has been made for convenience of the formulation, since as a rule it is met in applications. Theorem 4.1 would remain valid, however, with this assumption being replaced by a weaker one, by which $\{\Sigma^{\varepsilon} : 0 < \varepsilon \leqslant \varepsilon_0\}$ would be a family of C^r manifolds satisfying estimate (4.2).

Proof of Theorem 4.1. With Ψ defined as in Theorem 3.1, (a, b, x) satisfies (4.3) if and only if $a = \hat{a}(T)$, $b = \hat{b}(T)$, $x = \rho_{\varepsilon T}(x^0) + \hat{y}(T)$, where $(\hat{a}(\cdot), \hat{b}(\cdot), \hat{y}(\cdot)) = \Psi(a^1, b^0, x^0, T)$ solves (2.1), (2.2) (3.2), (3.12) with

$$b^0 = \sigma(\hat{a}(0), x^0) = \sigma(P_a \Psi(a, \sigma(\hat{a}(0), x^0), x^0, T)(0), x^0). \tag{4.7}$$

Equivalently, equality (4.7) means that $a^0 = \hat{a}(0)$ is a fixed point of the map $\mathcal{G} : \{|a^0| \leqslant \Delta\} \to \mathbb{R}^k$ with parameters x^0, T, $a^1 = a$ defined by

$$\mathcal{G}(a^0; a^1, x^0, T) = P_a \Psi(a^1, \sigma(a^0, x^0), x^0, T)(0).$$

We define

$$s(a, x^0, T) = P_{(b, x)} \Psi(a, \sigma(a^0, x^0), x^0, T)(T) \tag{4.8}$$

with a^0 a fixed point of $\mathcal{G}$. In order that s is well defined and C^r, we have to verify that this fixed point of $\mathcal{G}$ is unique and C^r.

To this end we employ the uniform contraction principle once more. Immediately from the definition of $\mathcal{G}$ we obtain $\mathcal{G}(0; a^1, x^0, T) = 0$. As a composition of the C^r functions σ, Ψ and the projection operator, $\mathcal{G}$ is C^r. From the chain rule [L] it follows that the derivatives of order j, $0 \leqslant j \leqslant r$ of $\mathcal{G}$ are polynomials of derivatives of order $\leqslant j$ of $P_a \Psi(.)(0)$ and σ, each term containing a derivative of $P_a \Psi(.)(0)$ as a factor. Therefore, the estimate (3.14) and the assumption (4.2) yield

$$\|D^j \mathcal{G}(a^0, a^1, x^0, T)\| = O(e^{-q/\varepsilon}) \tag{4.9}$$

for $0 \leqslant j \leqslant r$, $|a^1| \leqslant \Delta$, $x^0 \in J$, $a^0 \in \mathcal{E} : \{|a^0| \leqslant \Delta\}$.

Hence, for $0 < \varepsilon \leqslant \varepsilon_0$ sufficiently small, $\mathcal{G} : \mathcal{E} \to \mathcal{E}$ is a contraction uniform in $|a^1| \leqslant \Delta$, $x^0 \in J^{\varepsilon}$ and $T \in [\underline{\tau}/\varepsilon, \bar{\tau}/\varepsilon]$. By the uniform contraction principle it has a unique fixed point in $\mathcal{E}$ which is a C^r function of a^1, x^0, T. Thus, s is well defined by (4.8). The estimates (4.5), (4.6) follow immediately from (3.13)–(3.15) by formal differentiation of the fixed point identity and the chain rule formula. ▮

4.5. *Remark.* Note that, in addition to (4.6), Theorem 3.1 gives exponential closeness of the x-component of the trajectory through $(a, s(a, x^0, T))$ to $\rho_{\varepsilon t}(x^0)$ for all $0 \leqslant t \leqslant T$. Further, it gives $|a^0| = O(e^{-q/\varepsilon})$, so an exponentially small peace of σ (with respect to a) turns out to be relevant only.

In applications, it is often useful to have the invariant manifold parametrized as in Corollary 3.4, i.e., by natural local coordinates generated by its imbedding into the state space. In order to transform the parametrization of Theorem 4.1 into such a one non-tangency of the flow to the entry manifold Σ^ε is needed. Furthermore, Σ^ε is frequently given as the intersection of the invariant manifold with some face, e.g., $|b| = \Delta$ This ties the components x^0 and a^0, i.e., we cannot have them independent as in Theorem 4.1.

Hence, we define

$$\mathbf{\Sigma} = \{(a, b, x, \varepsilon) : b = \boldsymbol{\sigma}(a, x, \varepsilon), (a, x, \varepsilon) \in \mathbf{K}\},$$

where $\mathbf{K}$ is a C^r manifold of dimension $k + p$, $p \geqslant 0$, of the $(a, b, x, \varepsilon$-space having non-empty transversal intersections with the subspaces $\{a = \text{const}, \varepsilon = \text{const}\}$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and $|s| \leqslant \Delta$ for some $\Delta > 0$ and $\varepsilon_0 > 0$.

By transversality, the manifold $\mathbf{K}$ can be parametrized by a $k + p + 1$-dimensional parameter containing a and ε as components. A certain uniformity of the transversal intersection is needed: we assume that

$$\mathbf{K} = \{(a, x, \varepsilon) : |a| \leqslant \Delta, x = \boldsymbol{\kappa}(a, x^0, \varepsilon)\},$$

where $x^0 \in \mathbf{J}$, a submanifold of dimension p of $\mathbf{M}$, $\boldsymbol{\kappa}$ is C^r, $|\boldsymbol{\kappa}| \leqslant \Delta$ and its derivatives grow at most as powers of ε^{-1} as $\varepsilon \to 0$. As before, about $\boldsymbol{\sigma}$ we assume that it is C^r and satisfies (4.2).

As before, we denote the ε-sections of $\mathbf{K}$, $\mathbf{\Sigma}$, $\mathbf{J}$, $\boldsymbol{\kappa}$, $\boldsymbol{\sigma}$ by K^ε, Σ^ε, J^ε, $\boldsymbol{\kappa}$, σ^ε, respectively.

4.6. THEOREM. *Let* $\mathbf{\Sigma}$ *be defined by* (4.7) *and let* $\boldsymbol{\kappa}, \boldsymbol{\sigma}$ *satisfy the above assumption. Assume that* (3.41), (3.42) *hold true and that the map* $(x^\varepsilon, \tau) \to \rho^\varepsilon_\tau(x^\varepsilon)$ *is injective on* $J^\varepsilon \times [\underline{\tau}, \bar{\tau}]$. *Then, there is a* C^r *function* $\tilde{s}^\varepsilon : \rho_{[\underline{\tau}/\varepsilon, \bar{\tau}/\varepsilon]}(J^\varepsilon) \times \{|a| \leqslant \Delta\} \to \mathbb{R}^{l+m}$ *such that*

$$(a, b, x) = \varphi_T(a^0, b^0, x^0)$$

for some $(a^0, b^0, x^0) \in \Sigma^\varepsilon$ *and some* $\underline{\tau} \leqslant \varepsilon T \leqslant \bar{\tau}$ *if and only if*

$$(b, x) = \tilde{s}^\varepsilon(a^0, \xi) \qquad \text{and} \qquad \xi = \rho_{\varepsilon t}(x^0).$$

Moreover, one has

$$|D^j P_b \tilde{s}^\varepsilon(a, \xi)| = O(e^{-q/\varepsilon}) \tag{4.10}$$

$$|D^j P_x(\tilde{s}^\varepsilon(a, \xi) - \xi)| = O(e^{-q/\varepsilon}) \tag{4.11}$$

for $0 \leqslant j \leqslant r$ *and some* $q > 0$.

4.7. *Remark.* Theorem 4.6 is an exponential locally global C^r extension of [J, Lemma 6].

4.8 *Remark.* As in the case of Corollary 3.4, there is an obvious local version of Theorem 4.6 without the injectivity assumption of ρ (cf. Remark 3.5).

Proof of Theorem 4.6 The function $\tilde{s}$ is defined by

$$\tilde{s}(a, \xi) = P_{(b, x)} \varphi_{\tau/\varepsilon}(a^0, \sigma(a^0, \kappa(a^0, x^0)), \kappa(a^0, x^0), \tau/\varepsilon), \tag{4.12}$$

where a^0 solves the equation

$$a^0 = \Psi(a, \sigma(a^0, \kappa(a^0, x^0)), \kappa(a^0, x^0), \tau/\varepsilon)(0) \tag{4.13}$$

and $(x^0, \tau) = i(\xi)$ is as in (3.46). The map in the right-hand side of (4.13) is $\mathscr{G}$ of the proof of Theorem 4.1 with the argument (x^0, T) replaced by $i(\xi)$ and the argument x^0 of σ and Ψ replaced by $\kappa(a^0, x^0)$. The derivatives of both i and κ grow at most as powers of ε^{-1} with $\varepsilon \to 0$ while those of Ψ are exponentially small. Hence, as in the proof of Theorem 4.1 we obtain that

$$a^0 \mapsto \Psi(a, \sigma(a^0, \kappa(a^0, x^0)), \kappa(a^0, x^0), \tau/\varepsilon)(0)$$

is a contraction in $\{|a^0| \leqslant \Delta\}$ for $0 < \varepsilon \leqslant \varepsilon_0$ which is uniform in a, ξ. We conclude unique solvability of (4.13) and then obtain the estimates (4.10), (4.11) by the uniform contraction principle and the chain rule formula. Since there is no new idea compared to that proof, we omit the details. ∎

The injectivity assumption of Theorem 4.6 requires in fact the flow ρ^ε_τ to be parallel on the invariant manifold through J^ε. If, moreover, it is parallel on all M, Theorem 4.6 can be expressed in a particularly simple form, which corresponds to the original formulation of EL of [JK, JKK]: If ρ^ε_τ

is parallel in M for $0 \leqslant \varepsilon \leqslant \varepsilon_0$, then there exists a C^r change of variables $x \mapsto (\zeta, z)$. ($\zeta \in \mathbb{R}^1$, $z \in \mathbb{R}^{m-1}$) transforming (3.1) into the system of equations

$$\zeta' = 1$$
$$z' = 0$$

the flow of which is

$$\rho_\tau(\zeta, z) = (\zeta + \tau, z). \tag{4.14}$$

Assumption (3.41) transforms into

$$(1, 0) \notin T_{(\zeta, z)} J^\varepsilon \qquad \text{for} \quad (\zeta, z) \in J^\varepsilon.$$

Assumption (3.42) is satisfied if and only if J^ε is the graph of a C^r function $\gamma : P_z J^\varepsilon \to J^\varepsilon$, satisfying

$$|D\gamma| = O(\varepsilon^{-\lambda}). \tag{4.15}$$

Thus, we have

4.9. COROLLARY. *Let the assumptions of Theorem* 4.6 *be satisfied. In addition, assume that in suitable coordinates* (ζ, z) *in* M *the flow* ρ_τ *is expressed by* (4.14) *and* J^ε *is the graph of a* C^r *function* γ *satisfying* (4.15). *Then, there exists a* C^r *function* $\tilde{s} : \{|a| \leqslant \Delta\} \times \{\zeta \in (\underline{\tau}, \bar{\tau})\} \times Z \to \mathbb{R}^l \times \mathbb{R}^{m-1}$ *such that*

$$(a, b, \zeta, z) \in \varphi^\varepsilon_T(a^0, b^0, \zeta^0, z^0) \qquad \text{for} \quad \underline{\tau} \leqslant \varepsilon T < \bar{\tau}$$

for $(a^0, b^0, \zeta^0, z^0) \in \Sigma^\varepsilon$ *if and only if*

$$(b, z) = \tilde{s}(a, \zeta, z^0).$$

Moreover, $\tilde{s}$ *satisfies the estimates* $|\tilde{s}(a, \zeta, z^0) - (0, z)| = O(e^{-q/\varepsilon})$ *and* $|D^j_{(a, \zeta)} \tilde{s}| = O(e^{-q/\varepsilon})$ *for* $0 < j \leqslant r$ *for some* $q > 0$.

4.10. *Remark.* The original EL (called "(k+1)-EL" in [J]) has z^0 fixed. Then Σ^ε has dimension k and its invariant manifold has dimension $k+1$.

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